Beam Invariants For Diagnostics

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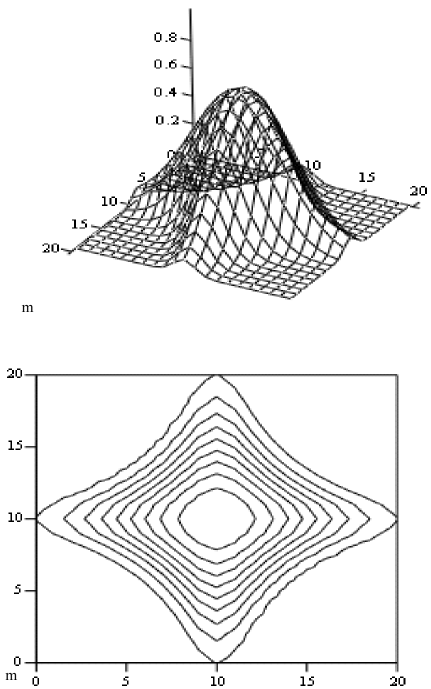
Motivation

1) 1D case universal invariant – normalized emittance.
   In linear case ellipse transforms into ellipse with the same area.
   In general nonlinear case area is preserved also (Liouville's theorem)
2) 2D, 3D cases – had no simple answer. Liouville's theorem still work (full volume is preserved), but more conditions should be met. Additional invariants will be presented
3) What if in 2 consequent nondestructive measurements invariants are not preserved? They are valid only for symplectic transformations (in nonrelativistic case – force depends on coordinates, in general – there exist canonical coordinate and momentum with Hamiltonian equations for them). Diffusion, friction, scattering, etc. spoil the invariants. Invariants behavior can tell us what kind of equation we need to use to describe the system.
4) Example – LEDA experiments don't show full agreement with simulation. Is it lack of our knowledge of the initial conditions or not accurate equations are used to simulate the beam?

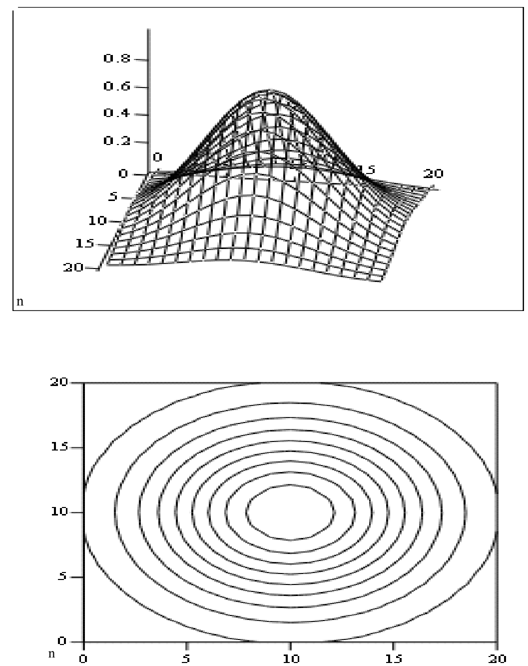
1D case

Ellipse equation -- $\gamma x^2+2\,\alpha x\,x'+\beta(x')^2=1$; **area** $\pi/\sqrt{\beta\gamma-\alpha^2}$ **is conserved.**

Beams can be transformed dynamically (symplectically) from one to another if and only if the area of cross section is the same at all the equivalent levels. The cross section levels are equivalent when the ratio of the particles above and below this level is the same.



1st DISTRIBUTION

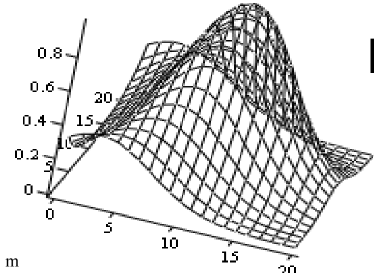


2nd DISTRIBUTION

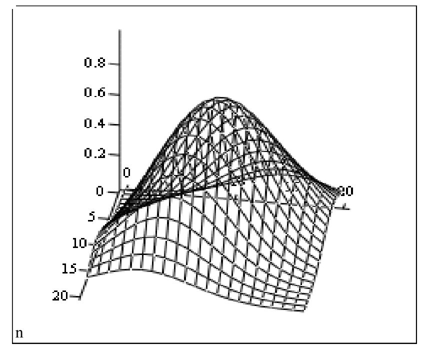
2D and 3D cases

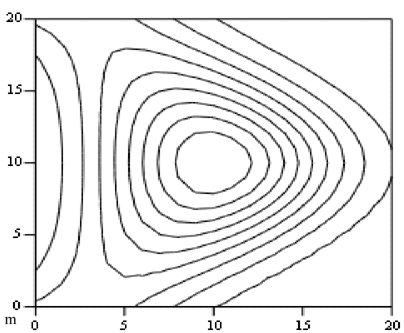
Basic facts of linear symplectic transformations: matrix M should satisfy

$M^T S M = S$ $\quad S = \begin{bmatrix} 0 & 1 & \dots \\ -1 & 0 & \dots \\ \dots & \dots & \dots \end{bmatrix}$ **1d case yields det M =1, 2d case - 5 more conditions, etc.**

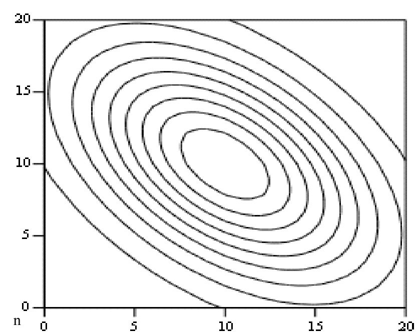


Maximum transforms into maximum
$\rightarrow$





XY projection around maximum has elliptical level lines

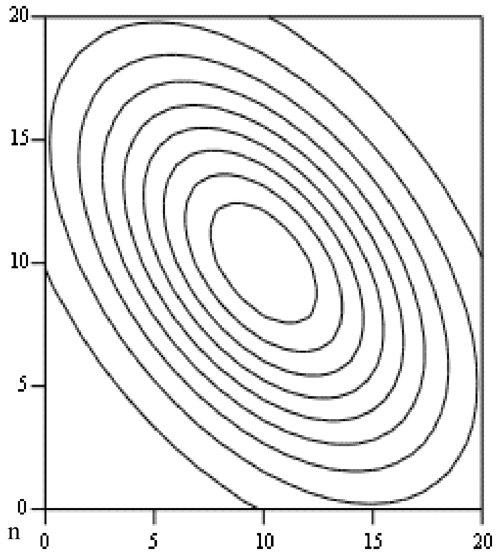


It is logical to solve the problem of equivalence for the particles around maximum (beam core) first

2D and 3D case- linearisation around maximum

Around maximum distribution has form f(x,y…)= $ax^2+2\,bx\,x'+2cxy+2dx\,y'+\ldots$

In matrix form $f=X^TQX$, where $Q=\begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}$ **for 2D case (taken as an example)**



Example of the a,b,c… coefficient measurements - get xy, xx'…distributions with other coordinates equal to zero– 6 projections in 2D case, 15 in 3D case

Necessary condition for linearized distributions 1 and 2 to be symplectically equivalent:

$S \bullet Q_1$ should have same eigenvalues with $S \bullet Q_2$.

Sufficient condition: these two similar matrices can be transformed one to another by some matrix T: $T^{-1}\, S \bullet Q_1\, T = S \bullet Q_2$. Matrix T should be symplectic.

Eigenvalues of the matrix S•Q are beam invariants!

2D case interpretation of found invariants

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In matrix form $f=X^TQX$, where $Q=\begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}$ **for 2D case (taken as an example)**

$$\mathrm{Det}(S\bullet Q-\lambda E)=\lambda^4-\mathrm{Tr}(S\bullet Q)\,\lambda^3+(ae-b^2+jh-i^2+2(gc-fd))\,\lambda^2+\mathrm{Coef}\,\lambda^1+\det(Q)=0$$

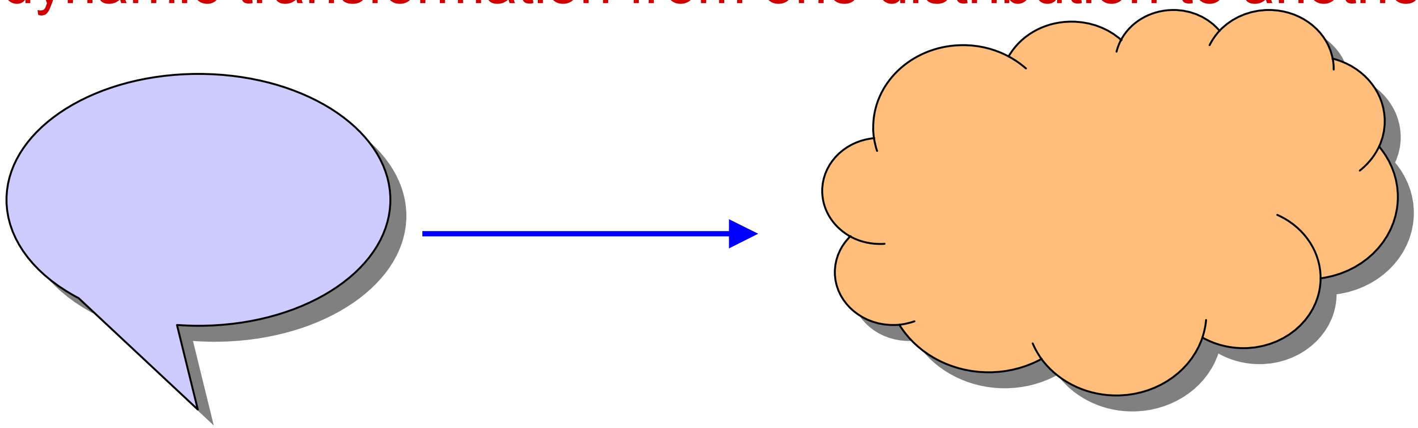
1) $\mathrm{Tr}(S\bullet Q)=0$ – no invariant condition
2) $ae-b^2+jh-i^2+2(gc-fd)=1/\varepsilon_x^2+1/\varepsilon_y^2+2/\varepsilon_c^2$ = invariant –sum of inversed square emittances plus doubled inversed "coupling" emittance squared is invariant!
3) Coef of λ^1 = inv – physical meaning unknown
4) Det(Q) = inv – phase volume conservation (Liouville's theorem)

These invariants valid for beam core elliptic coefficients, and for rms emittances, if the tails are small. If tails are big – rms emittance is a poor characterization of the beam.

Additional remark – canonical variables should be used for coefficients determination. If the energy change is large between measurements, transverse momenta should be used instead of transverse angles. If the magnetic field in the measurement region is strong, canonical momenta are conventional momenta plus $\frac{e}{c}\vec{A}$ (A is the vector potential)

Search for General Case Invariants

- With two given arbitrary beam distributions, is it possible to find dynamic transformation from one distribution to another



- There exist invariants for the phase space distributions that should be conserved. One first invariant – the phase space volume. Others are not determined yet.

Summary

- Beam core invariants found for 2D-3D cases
- They can serve as a good beam core characterization
- Their nonconservation in experiments may lead to modification of the equation for the beam simulation
- Invariants form is not trivial but conceivable, the physical meaning of all invariants not fully understood
- To get full invariants all correlations (xy’, yx’, etc.) should be measured
- For the beam halo the problem is formulated, but full answers are not obtained yet